Optimizing Age of Information with Attacks

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Abstract. This paper provides an analysis of the impact of attacks on the Age of Information (AoI) in queuing systems. We consider single and tandem queue models, with and without preemption in service. We have shown that when preemption in service is allowed, attacks do not result in AoI reduction. For non preemptive servers, we show that attacks can reduce the AoI for both tandem and single server cases. For the single server case, we establish necessary and sufficient conditions for the existence of arrival and attacks rates that minimize the AoI and provide an explicit expression of the optimal attack rate when the arrival rate of updates is very large. Numerical results corroborate the analytical claims and show the accuracy of the obtained results when the arrival rate of update is low.

1 Introduction

This paper deals with a scenario in which a process needs to be observed remotely in such a way that the information about the status of this process must be as recent as possible. Such a situation occurs in a wide range of applications, being the most popular the autonomous driving systems in which recent traffic information is crucial to take adequate decisions. The Age of Information (AoI) is a novel performance metric that measures the freshness of the information about the status of a remote system. Since its introduction in the seminal papers [3, 4], the main goal has been to characterize the average AoI of systems (or other alternative performance metrics such as the Peak Age of Information, the Query Age of Information or the Age of Incorrect Information) as well as to study the conditions under which the AoI can be reduced. In general, the channel/environment between the source and remote monitor has been generally modeled as a queueing system, which can capture the impact of transmission delay due to the fact that the remote receiver is far away from the source. We refer to [9] for a recent survey on this topic.

We consider a system with attacks. These attacks provoke that all the updates in the queues are discarded, i.e., that the queueing system gets empty. They model, for instance, the behavior of a malicious adversary that aims to avoid that the monitor receives information about the status of the process of interest. The attacks under study here have a clear negative effect if we consider other performance metrics widely studied in queueing models such as throughput. In this work, we show that the presence of attacks can improve performance when the metric under consideration is the AoI. More precisely, we show that for a single server and for a tandem queue system, when we allow preemption in service, the AoI is not reduced by the presence of attacks. However, for the preemptive case, the AoI might be reduced due to the presence of attacks.

Furthermore, for a non preemptive single server, we provide the following analytical results. First, we establish necessary and sufficient conditions for the existence of an arrival rate of attacks such that the average AoI is reduced. Then, we provide an analytical expression of the arrival rate of attacks that minimizes the average Age of Information when the arrival rate of updates is much larger than the service rate. Finally, we show that when the arrival rate of updates tends to infinity, $\mathcal{R}$, which is the maximum reduction factor due to the presence of attacks, tends to 2.

The authors in [1] consider an M/M/1/2 queue and analyze the average AoI when there is a deterministic or exponentially distributed deadline in the packet that is waiting in the queue. More precisely, in their model, the packet that is waiting is discarded when its sojourn exceed its deadline. Our work aims to study a more general framework by considering that the discarded packets are, not only those that are waiting in the queue, but also the packets that are in service. Our work is also close to [5], where the M/M/1/1 queue with preemption and without preemption is considered and the server changes from On to Off state with exponential times. When the server changes from On to Off and it is busy, it stops serving packets and, when it changes from Off to On again, the packet that has been stopped is resumed. In our model, however, when an attack occurs, packets in service are discarded (i.e., they are not served again). Another difference of our model is that we consider not only single servers but also tandem queues.

2 Model Description

We consider a system in which the status of one process is observed remotely by a monitor. The transmission channel/medium through which status updates are sent to the monitor is modeled as a queueing system. We assume that the transmission times from the sources to the transmission channel and from the transmission channel to the monitor are both zero. Therefore, we use interchangeably the terms generation time of updates and arrival time of updates to the queueing system, as well as the terms end of service at the queueing system and delivery to the monitor.

We consider that an attack occurs following a Poisson process of rate α. When this occurs, all the updates in the transmission channel are discarded, i.e., the transmission channel gets empty. We assume that the system can operate normally right after an attack (i.e., the transmission channel does not need any recovery time after an attack).

As a metric of performance of the model under study, we consider the AoI, which is defined as the time elapsed since the generation time of the last status update that has been delivered to the monitor. Our goal is to analyze the influence of the attacks on the AoI. Thus, we denote by $\Delta(\alpha)$ the average Age of Information (AAoI) when the rate at which attacks occurs is α. For the models under study, we first study α^*, which is defined as the value of α that minimizes $\Delta(\alpha)$. Furthermore, we investigate the performance improvement due to the presence of attacks. For this purpose, we define $\mathcal{R}$ as the ratio between the AAoI when $\alpha = 0$ and the AAoI for the value of α that minimizes Δ, i.e.,

$$\mathcal{R} = \frac{\lim_{\alpha \to 0^+} \Delta(\alpha)}{\min_\alpha \Delta(\alpha)}. \tag{1}$$

3 Single Queues

3.1 Preemptive Queues

We consider a single queue. Updates arrive to the system following a Poisson process of rate λ and are served with exponential time with rate μ. We consider that preemption in service is allowed, i.e., when an incoming update arrives to the queue and the server is busy, the incoming update replaces the current update in service.

The authors in [8, Thm 2a)] characterize the AAoI of source i (i.e., the source of interest) in a M/M/1/1 queue with preemption in service and where there are other sources that are not of interest also send updates to the queue. In their model, an arrival from an update of a source which is not of the source of interest replaces the packet in service. Besides, the end of the service of a packet which is not of the source of interest do not modify the AoI of the source of interest. Therefore, since there is at most one packet in the system, replacing an update in service by a packet which is not from the source of interest has the same effect for the AoI as removing the packet in service (which occurs when an attack occurs); in both cases, the AoI is not modified until the end of the service of a new incoming packet of the source of interest. Hence, we conclude that the AAoI of a single queue with preemption in service coincides with that of [8, Thm 2a)]. As a result, we have that the AAoI of a M/M/1/1 queue with preemption in service and attacks is

$$\Delta(\alpha) = \frac{1}{\lambda}\left(1 + \frac{\lambda + \alpha}{\mu}\right).$$

We observe that $\Delta(\alpha)$ is clearly an increasing function of α. Therefore, we conclude the attacks do not reduce the AAoI for this model. As we will see next, this is not the case when preemption is not allowed.

3.2 Non Preemptive Queues

We consider a single queue with Poisson arrivals of rate λ and exponential service times with rate μ in which, unlike in the previous model, we do not allow preemption in service, i.e., when an incoming update arrives to the queue and the server is busy, the incoming update is discarded.

The author in [7, Section IIIA] characterizes the AAoI single queue without preemption and abandonment. We note that, since there is at most one update in this model, an abandonment and an attack have the same effect in the system, i.e., in both cases, the server gets idle. Therefore, we conclude that our model coincides with that of [7, Section IIIA]. As a consequence, from the result of [7, Section IIIA], it follows that the AAoI of a system of a M/M/1/1 queue without preemption and with attacks is

$$\Delta(\alpha) = \frac{1}{\lambda} + \frac{1}{\mu} + \frac{\lambda}{(\mu+\alpha)(\lambda+\mu+\alpha)} + \frac{\alpha}{\lambda\mu}. \tag{2}$$

The author in [7, Section IIIA] says that, when λ/μ is sufficiently large, a positive rate of abandonments (in our model, attacks) leads to an improvement of the AAoI, i.e., the AAoI is a decreasing function of α when $\alpha \to 0$. Here, we study analytically the value of α that minimizes (2). We first show that (2) is convex in α.

Lemma 1. (2) *is convex in* α.

Proof. The derivative of (2) with respect to α is:

$$\frac{1}{\lambda\mu} - \frac{\lambda(\lambda+2(\alpha+\mu))}{(\alpha+\mu)^2(\lambda+\alpha+\mu)^2}. \tag{3}$$

We compute the derivative with respect to α of the above expression and it results:

$$\frac{-2\lambda(\alpha+\mu)^2(\lambda+\alpha+\mu)^2}{(\alpha+\mu)^4(\lambda+\alpha+\mu)^4} + \frac{2\lambda(\alpha+\mu)(\lambda+\alpha+\mu)(\lambda+2(\alpha+\mu))(\lambda+\alpha+\mu+\alpha+\mu)}{(\alpha+\mu)^4(\lambda+\alpha+\mu)^4}.$$

The above expression is positive if and only if

$$-(\alpha+\mu)(\lambda+\alpha+\mu) + (\lambda+\alpha+\mu+\alpha+\mu)^2 > 0.$$

And the above expression clearly holds since

$$\begin{aligned}&(\lambda+\alpha+\mu+\alpha+\mu)^2 = (\lambda+\alpha+\mu)^2 + (\alpha+\mu)^2 + 2(\lambda+\alpha+\mu)(\alpha+\mu)\\&> (\lambda+\alpha+\mu)(\alpha+\mu).\end{aligned}$$

Since α is positive, using the above result, we know that the global minimum of (2) is strictly positive when its decreasing at $\alpha = 0$. We now provide a necessary and sufficient condition for this fact.

Lemma 2. (2) *is decreasing at* $\alpha = 0$ *if and only if* $\frac{\lambda}{\mu} > 1.2469$.

Proof. The derivative with respect to α of (2) is given in (3), which at $\alpha = 0$ equals $\frac{1}{\lambda\mu} - \frac{\lambda(\lambda+2\mu)}{\mu^2(\lambda+\mu)^2}$. This expression is negative if and only if

$$\lambda^2(\lambda + 2\mu) > \mu(\lambda + \mu)^2.$$

Let $\rho = \lambda/\mu$. The above expression is equivalent to:

$$\rho^2(\rho + 2) > (\rho + 1)^2 \iff \rho^3 + \rho^2 - 2\rho - 1 > 0.$$

We now notice that $\rho^3 + \rho^2 - 2\rho - 1 = 0$ has a unique positive root, which is 1.2469. And the desired result follows since $\rho^3 + \rho^2 - 2\rho - 1$ is positive when $\rho > 1.2469$.

From the above results, we conclude that the optimal rate of attacks is positive if and only if $\lambda > 1.2469\mu$.

Proposition 1. *The rate of attacks that minimizes* (2) *is strictly positive if and only if* $\frac{\lambda}{\mu} > 1.2469$.

In Proposition 1, we provide necessary and sufficient conditions such that the rate of attacks that minimizes AAoI is positive. The derivation of an analytical expression of the value of α^* (i.e., the value of α that minimizes (2)) requires to compute the roots of (3), which is equivalent to find the roots of a polynomial of degree four. α^* can hence be obtained numerically in a general setting. However, we now focus on $\lambda >> \mu$ and we study the value of α^* for this case.

Proposition 2. *When* $\lambda >> \mu$*, the value of* α *that minimizes* (2) *is* $\alpha^* \approx \sqrt{\mu}(\sqrt{\lambda} - \sqrt{\mu})$.

Proof. Let $\lambda = C\mu$. We show that the desired result follows when $C \to \infty$. For $\lambda = C\mu$, the derivative with respect to α of (2) is zero if and only if

$$\frac{1}{C\mu^2} - \frac{C\mu(C\mu + 2(\alpha + \mu))}{(\alpha + \mu)^2(C\mu + \alpha + \mu)^2} = 0 \iff \frac{C^2\mu^3(C\mu + 2(\alpha + \mu))}{(\alpha + \mu)^2(C\mu + \alpha + \mu)^2} = 1.$$

When $C \to \infty$, we have that

$$\frac{C^2\mu^3(C\mu + 2(\alpha + \mu))}{(\alpha + \mu)^2(C\mu + \alpha + \mu)^2} \approx \frac{C\mu^2}{(\alpha + \mu)^2}.$$

Therefore, for $C \to \infty$, the derivative with respect to α of (2) is zero when

$$\frac{C\mu^2}{(\alpha + \mu)^2} \approx 1 \iff \alpha^* \approx \mu(\sqrt{C} - 1) = \sqrt{\mu}(\sqrt{\lambda} - \sqrt{\mu}),$$

where the last equality follows since $\lambda = C\mu$, i.e., $C = \lambda/\mu$.

From the above result, we conclude that, when $\lambda \to \infty$, the value of α that minimizes (2) tends to infinity as well. We now focus on $\mathcal{R}$, which, according to (1) and (2), gives for this model

$$\mathcal{R} = \frac{\frac{1}{\lambda} + \frac{1}{\mu} + \frac{\lambda}{\mu(\lambda+\mu)}}{\min_\alpha \left(\frac{1}{\lambda} + \frac{1}{\mu} + \frac{\lambda}{(\mu+\alpha)(\lambda+\mu+\alpha)} + \frac{\alpha}{\lambda\mu} \right)}.$$

For the denominator, we have from Proposition 2 that, when $\lambda >> \mu$,

$$\min_\alpha \left(\frac{1}{\lambda} + \frac{1}{\mu} + \frac{\lambda}{(\mu+\alpha)(\lambda+\mu+\alpha)} + \frac{\alpha}{\lambda\mu} \right) =$$
$$\frac{1}{\lambda} + \frac{1}{\mu} + \frac{\lambda}{(\mu+\sqrt{\mu}(\sqrt{\lambda}-\sqrt{\mu}))(\lambda+\mu+\sqrt{\mu}(\sqrt{\lambda}-\sqrt{\mu}))} + \frac{\sqrt{\mu}(\sqrt{\lambda}-\sqrt{\mu})}{\lambda\mu} =$$
$$\frac{1}{\mu} + \frac{1}{\sqrt{\mu}(\sqrt{\lambda}+\sqrt{\mu})} + \frac{1}{\sqrt{\lambda\mu}}.$$

As a result, when $\lambda \to \infty$, the denominator of $\mathcal{R}$ tends to $1/\mu$. Furthermore, the numerator of $\mathcal{R}$ tends to $2/\mu$ when $\lambda \to \infty$. Thus, the next result follows.

Proposition 3. *When $\lambda \to \infty$, $\mathcal{R} \to 2$.*

4 Tandem Queues

4.1 Preemptive Queues

We consider a system with n tandem queues. The service rate of queue i is exponentially distributed with rate $\mu_i > 0$, $i = 1, 2, \ldots, n$. Updates arrive from outside to Server 1 and we assume that the generation times follow a Poisson process of rate λ. For $i = 1, \ldots, n-1$, when an update in Server i ends service, it is immediately sent to Server $i+1$. We allow preemption of updates in service, i.e., when an update arrives to a server that is busy, the incoming update starts service and the update that was being served is lost. We provide an explicit expression of the AAoI for this model. The proof is given in A.

Proposition 4. *The AAoI of a system of n tandem preemptive servers with attacks is*

$$\Delta(\alpha) = \frac{1}{\lambda} + \sum_{i=1}^{n} \frac{1}{\mu_i} \prod_{j=0}^{i-1} \left(1 + \frac{\alpha}{\mu_j}\right). \tag{4}$$

We observe that the AAoI is a polynomial of degree n with positive coefficients. This implies that $\alpha^* = 0$ for this model and, as a a result, we have that $\mathcal{R} = 1$. Therefore, we conclude the attacks do not reduce the AAoI for this model.

4.2 Two Non Preemptive Queues

We consider a system with two tandem queues with Poisson arrivals with rate λ and exponential service times with rate μ_i at Server i. Servers are non preemptive, i.e., when an update arrives to a server that is busy, the incoming update is discarded. The AAoI of this system without attacks has been studied in [2]. We aim to study whether the presence of attacks reduces the AAoI. For this purpose, we model the system using the SHS technique. In Figure 1 we consider $\mu_1 = \mu_2 = 1$ and, for different values of λ, we plot the AAoI of this system when α changes from 0.1 to 5. We observe that, for $\lambda = 1$ and $\lambda = 10$, we have that $\alpha^* > 0$, which implies that $\mathcal{R} > 1$, i.e., the attacks can reduce the AAoI in this system.

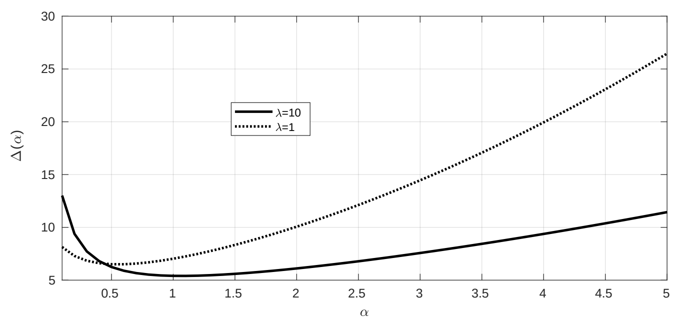

Fig. 1: AAoI with respect to α for $\mu_1 = \mu_2 = 1$ and different values of λ.

5 Numerical Experiments

The result of Proposition 2 can be used to provide an approximation of the value of α^* for an arbitrary λ. In Table 1, we study the accuracy of this approximation for different values of λ and $\mu = 1$. We observe that the value of $\sqrt{\mu}(\sqrt{\lambda} - \sqrt{\mu})$ is close to α^* even for λ small.

References

1. C. Kam, S. Kompella, G. D. Nguyen, J. E. Wieselthier, and A. Ephremides. On the age of information with packet deadlines. *IEEE Transactions on Information Theory*, 64(9):6419–6428, 2018.
2. C. Kam, J. P. Molnar, and S. Kompella. Age of information for queues in tandem. In *MILCOM 2018 - 2018 IEEE Military Communications Conference (MILCOM)*, pages 1–6, 2018.

	α^*	$\sqrt{\mu}(\sqrt{\lambda}-\sqrt{\mu})$
$\lambda = 3$	0.6219	0.732
$\lambda = 10$	1.133	1.236
$\lambda = 30$	4.413	4.477
$\lambda = 50$	6.017	6.071
$\lambda = 100$	8.959	9
$\lambda = 200$	13.111	13.142

Table 1: Comparison of α^* and the approximation based on Proposition 2 for different values of λ and $\mu = 1$.

3. S. Kaul, M. Gruteser, V. Rai, and J. Kenney. Minimizing age of information in vehicular networks. In *2011 8th Annual IEEE Communications Society Conference on Sensor, Mesh and Ad Hoc Communications and Networks*, pages 350–358. IEEE, 2011.
4. S. Kaul, R. Yates, and M. Gruteser. Real-time status: How often should one update? In *2012 Proceedings IEEE INFOCOM*, pages 2731–2735. IEEE, 2012.
5. A. Sinha, P. D. Mankar, N. Pappas, and H. S. Dhillon. Age of information with on-off service. In *2023 IEEE Information Theory Workshop (ITW)*, pages 445–450. IEEE, 2023.
6. R. D. Yates. Age of information in a network of preemptive servers. In *IEEE INFOCOM 2018-IEEE Conference on Computer Communications Workshops (INFOCOM WKSHPS)*, pages 118–123. IEEE, 2018.
7. R. D. Yates. The age of information in networks: Moments, distributions, and sampling. *IEEE Transactions on Information Theory*, 66(9):5712–5728, 2020.
8. R. D. Yates and S. K. Kaul. The age of information: Real-time status updating by multiple sources. *IEEE Transactions on Information Theory*, 65(3):1807–1827, 2018.
9. R. D. Yates, Y. Sun, D. R. Brown, S. K. Kaul, E. Modiano, and S. Ulukus. Age of information: An introduction and survey. *IEEE Journal on Selected Areas in Communications*, 39(5):1183–1210, 2021.

A Proof of Proposition 4

We model the system witn n tandem preemptive servers using the SHS technique [8]. The continuous state is a vector $\mathbf{x}(t) = [x_0(t)\, x_1(t)\, \ldots\, x_n(t)]$ and the Markov chain is formed by a single state. Besides, $x_0(t)$ represents the age at the monitor and, for $i = 1, \ldots, n$, $x_i(t)$ represents the generation time of the update in Server i when Server i is busy and, when an update is sent from Server i to the next one, it is the timestamp of a fake update that is put in Server i and does not modify the age of the system.

Here, we consider the SHS derivation [6] with an additional transition which is related to the attacks. For more details on the SHS technique and derivation, the reader can refer to [6]. Thus, the SHS Markov chain we consider has also a single state and the table of transitions of the SHS method is presented in Table 2. We explain each transition now.

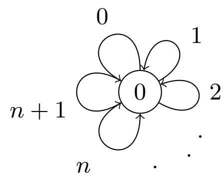

Fig. 2: The SHS Markov chain for n tandem preemptive servers with attacks.

l	$\lambda^{(l)}$	$\mathbf{x}\mathbf{A_l}$	$\bar{\mathbf{v}}_{\mathbf{q_l}}\mathbf{A_l}$
0	λ	$[x_0\ 0\ x_2\ x_3\ \dots\ x_{n-1}\ x_n]$	$[\bar{v}_{0,0}\ \ 0\ \ \bar{v}_{0,2}\ \bar{v}_{0,3}\ \dots\ \bar{v}_{0,n-1}\ \bar{v}_{0,n}]$
1	μ_1	$[x_0\ x_1\ x_1\ x_3\ \dots\ x_{n-1}\ x_n]$	$[\bar{v}_{0,0}\ \bar{v}_{0,1}\ \bar{v}_{0,1}\ \bar{v}_{0,3}\ \dots\ \bar{v}_{0,n-1}\ \bar{v}_{0,n}]$
2	μ_2	$[x_0\ x_1\ x_2\ x_2\ \dots\ x_{n-1}\ x_n]$	$[\bar{v}_{0,0}\ \bar{v}_{0,1}\ \bar{v}_{0,2}\ \bar{v}_{0,2}\ \dots\ \bar{v}_{0,n-1}\ \bar{v}_{0,n}]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	μ_{n-1}	$[x_0\ x_1\ x_2\ x_3\ \dots\ x_{n-1}\ x_{n-1}]$	$[\bar{v}_{0,0}\ \bar{v}_{0,1}\ \bar{v}_{0,2}\ \bar{v}_{0,3}\ \dots\ \bar{v}_{0,n-1}\ \bar{v}_{0,n-1}]$
n	μ_n	$[x_n\ x_1\ x_2\ x_3\ \dots\ x_{n-1}\ x_n]$	$[\bar{v}_{0,n}\ \bar{v}_{0,1}\ \bar{v}_{0,2}\ \bar{v}_{0,3}\ \dots\ \bar{v}_{0,n-1}\ \bar{v}_{0,n}]$
$n+1$	α	$[x_0\ x_0\ x_0\ x_0\ \dots\ x_0\ x_0]$	$[\bar{v}_{0,0}\ \bar{v}_{0,0}\ \bar{v}_{0,0}\ \bar{v}_{0,0}\ \dots\ \bar{v}_{0,0}\ \ \bar{v}_{0,0}]$

Table 2: Table of transitions of the SHS method for n tandem preemptive servers with attacks.

- For $l = 0$, a new update arrives to Server 1 and, since the update in Server 1 is preempted, the value of the second element of $\mathbf{x}$ is replaced by zero, whereas the value of the rest of the elements of $\mathbf{x}$ is not modified.
- For $l \in \{1, \dots, n-1\}$, an update in Server l ends service and is sent to Server $l+1$. Therefore, the value of the $l+2$-th element of $\mathbf{x}$ is replaced by that of the $l+1$-th element and, since we put a fake update in Server l with the same timestamp as the update that has just ended its service, the value of the $l+1$-th element of $\mathbf{x}$ is not modified. The rest of the elements of $\mathbf{x}$ are not modified.
- For $l = n$, the update of Server n ends its service and it is sent to the monitor. As a result, the value of the first element of $\mathbf{x}$ changes to that of the last element and the last element of $\mathbf{x}$ does not change since we put a fake update in Server n. The rest of the elements of $\mathbf{x}$ are not modified.
- For $l = n + 1$, an attack arrives and all the updates are discarded, but a fake update with the same timestamp as that of the monitor is put in all the servers. Therefore, all the elements of $\mathbf{x}$ changes to x_0, which is the age of the monitor.

Since the Markov chain has a single state, the stationary distribution is clearly $\pi_q = 1$. The age of the real or fake updates increases at unit rate and, therefore, $\mathbf{b}_q$ is a vector formed by $n+1$ ones. In the following, we use the notation $\mu_0 = \lambda$ and $\mu_{n+1} = \alpha$ to simplify the presentation. We apply [8, Thm 4] for the transitions of Table 2 and we obtain the following expressions:

$$\bar{v}_{0,0}\left(\sum_{i=0}^{n+1}\mu_i\right) = 1 + \bar{v}_{0,0}\left(\sum_{i=0}^{n-1}\mu_i\right) + \bar{v}_{0,n}\mu_n + \bar{v}_{0,0}\mu_{n+1} \tag{5}$$

$$\bar{v}_{0,1}\left(\sum_{i=0}^{n+1}\mu_i\right) = 1 + \bar{v}_{0,1}\left(\sum_{i=}^{n}\mu_i\right) + \bar{v}_{0,0}\mu_{n+1} \tag{6}$$

$$\bar{v}_{0,k}\left(\sum_{i=0}^{n+1}\mu_i\right) = 1 + \bar{v}_{0,k}\left(\sum_{i=0}^{k-1}\mu_i\right) + \bar{v}_{0,k-1}\mu_{k-1} + \bar{v}_{0,k}\left(\sum_{i=k}^{n}\mu_i\right), \quad k = 2, 3, \ldots, n. \tag{7}$$

Simplifying (5), we get

$$\bar{v}_{0,0} = \frac{1}{\mu_n} + \bar{v}_{0,n}, \tag{8}$$

whereas from (6) that

$$\bar{v}_{0,1}\left(1 + \frac{\mu_{n+1}}{\mu_0}\right) = \frac{1}{\mu_0} + \bar{v}_{0,0}\frac{\mu_{n+1}}{\mu_0}, \tag{9}$$

and from (7) that, if $k = 2, \ldots, n$, then

$$\bar{v}_{0,k}\left(1 + \frac{\mu_{n+1}}{\mu_{k-1}}\right) = \frac{1}{\mu_{k-1}} + \bar{v}_{0,k-1} + \frac{\mu_{n+1}}{\mu_{k-1}}\bar{v}_{0,0}, \tag{10}$$

Let $B_n = \prod_{i=0}^{n}\left(1 + \frac{\mu_{n+1}}{\mu_i}\right)$. We multiply both sides of (8) by B_{n-1} we get:

$$\begin{aligned}\bar{v}_{0,0}B_{n-1} &= \frac{B_{n-1}}{\mu_n} + \bar{v}_{0,n}B_{n-1} = \frac{B_{n-1}}{\mu_n} + \bar{v}_{0,n}\left(1 + \frac{\mu_{n+1}}{\mu_{n-1}}\right)B_{n-2} \\ &= \frac{B_{n-1}}{\mu_n} + \left(\frac{1}{\mu_{n-1}} + \bar{v}_{0,n-1} + \bar{v}_{0,0}\frac{\mu_{n+1}}{\mu_{n-1}}\right)B_{n-2},\end{aligned}$$

where in the second equality we use that $B_{n-1} = \left(1 + \frac{\mu_{n+1}}{\mu_{n-1}}\right)B_{n-2}$ and in the third equality we use (10) with $k = n$. Since $B_{n-1} = B_{n-2} + \frac{\mu_{n+1}}{\mu_{n-1}}B_{n-2}$, we note that the term $\frac{\mu_{n+1}}{\mu_{n-1}}B_{n-2}\bar{v}_{0,0}$ appears on both sides of the above expression and, therefore, both terms are canceled. Hence, we get $\bar{v}_{0,0}B_{n-2} = \frac{B_{n-1}}{\mu_n} + \frac{B_{n-2}}{\mu_{n-1}} + B_{n-2}\bar{v}_{0,n-1}$. Applying this arguments recursively, it results that

$$\bar{v}_{0,0}B_1 = \frac{B_{n-1}}{\mu_n} + \frac{B_{n-2}}{\mu_{n-1}} + \cdots + \frac{B_0}{\mu_1} + B_0\bar{v}_{0,1},$$

which using (9) gives

$$\bar{v}_{0,0}B_1 = \frac{B_{n-1}}{\mu_n} + \frac{B_{n-2}}{\mu_{n-1}} + \cdots + \frac{B_0}{\mu_1} + \frac{1}{\mu_0} + \bar{v}_{0,0}\frac{\mu_{n+1}}{\mu_0}.$$

Since $\bar{v}_{0,0}\frac{\mu_{n+1}}{\mu_0}$ appears on both sides of the last expression, the above formula is equivalent to

$$\bar{v}_{0,0} = \frac{B_{n-1}}{\mu_n} + \frac{B_{n-2}}{\mu_{n-1}} + \cdots + \frac{B_0}{\mu_1} + \frac{1}{\mu_0}.$$